Abstract vector spaces

The only vector space that we've seen so far is $\mathbb{R}^n$. In $\mathbb{R}^n$, we can add two vectors, and take a scalar multiple of a vector. We can generalize these and get a much broader collection of vector spaces.

Definition: A vector space consists of a nonempty set V of vectors that can be added and multiplied by a real number (a scalar), and for which the following axioms hold:

Addition

A1. If $\vec{u}$ and $\vec{v}$ are in V, then so is $\vec{u}+\vec{v}$.

A2. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for all $\vec{u}$ and $\vec{v}$ in V.

A3. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for $\vec{u},\vec{v},\vec{w}$ in V.

A4. An element $\vec{0}$ in V exists such that $\vec{v}+\vec{0} = \vec{v}$ for every $\vec{v}$.

A5. For each $\vec{v}$, there is a $-\vec{v}$ in V such that

$$\vec{v}+(-\vec{v}) = \vec{0}.$$

Scalar multiplication

S1. If $\vec{v}$ is in V, then so is $a\vec{v}$ for all real a.

S2. $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$.

S3. $(a+b)\vec{v} = a\vec{v} + b\vec{v}$.

S4. $a(b\vec{v}) = (ab)\vec{v}$.

S5. $1\vec{v} = \vec{v}$ for all $\vec{v}$ in V.

Note that $\mathbb{R}^n$ satisfies all these axioms.

Ex: The set M_{mn} of all $m \times n$ matrices is a vector space using matrix addition and scalar multiplication.

Ex: Any subspace U of $\mathbb{R}^n$ is a vector space:

Recall that U contains $\vec{0}$ and is closed under addition and scalar multiplication. Those are axioms A1, A4, S1.

U satisfies A2, A3, S2-S5 because $\mathbb{R}^n$ does.

For A5., if $\vec{v}$ is in U, then $(-1)\vec{v} = -\vec{v}$ is as well.

Vector space of polynomials

A <u>polynomial</u> in x is an expression

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

$a_0, a_1, \ldots, a_n$ are real numbers (called coefficients).

The highest power of x with nonzero coefficient is the <u>degree</u> of p, and its coeff is the <u>leading coefficient</u>.

So $\deg(1+4x+3x^3)=3$, $\deg(4)=0$.

Note that we can add two polynomials:

$$(1+3x)+(4x+2x^4) = 1+7x+2x^4.$$

We can also take the scalar product:

$$-7(1+x^3) = -7-7x^3$$

The zero polynomial is just 0. The rest of the axioms are easily checked.

So the set of all polynomials, P, is a vector space.

<u>Ex</u>: Let P_n be the set of all polynomials of degree at most n, together with the zero polynomial.

Sums and scalar multiples of polynomials in P_n are still in P_n. The rest of the axioms are inherited from P, so P_n is a vector space.

Ex: What about polynomials of degree at least n?
e.g. let V be polynomials of degree ≥ 2, together w/ 0

Then $p(x) = x + x^2$, $q(x) = x^2$ have degree 2, so are both in V, but $p(x) - q(x) = x$, which has degree 1, so it's not in V.

Thus V doesn't satisfy A1, so it's not a vector space.

Vector space of functions

If a and b are real numbers and $a < b$, the interval $[a, b]$ is all real #s x such that $a \leq x \leq b$.

A real-valued function f on $[a, b]$ is a function that takes as input values in $[a, b]$, and outputs a real #.

e.g. $f(x) = 2^x$, $f(x) = \sin x$.

$f(x) = \frac{1}{1+x}$ is a real-valued function on $[0, 1]$.

The set of all functions on $[a, b]$ is denoted $F[a, b]$.

If f and g are two functions, and r a real #, define

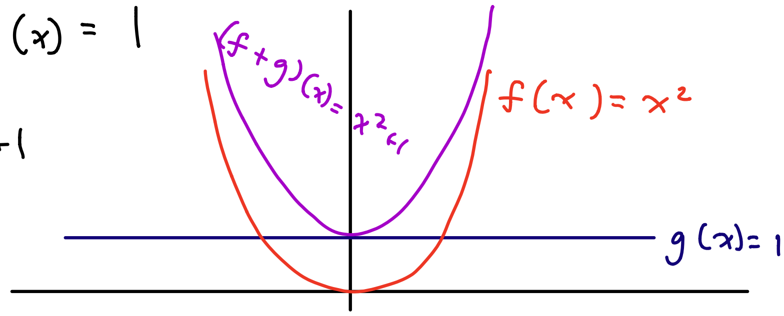
$$(f+g)(x) = f(x) + g(x)$$

$$(rf)(x) = r(f(x)).$$

This is called pointwise addition and scalar multiplication.

Ex: $f(x) = x^2$, $g(x) = 1$

$\Rightarrow (f+g)(x) = x^2+1$



$F[a,b]$ satisfies all the axioms:

A1 and S1 by the above.

A2, A3, S2-S5 because $\mathbb{R}$ does.

For A4, we define the zero function: $0(x) = 0$.

Then $(f+0)(x) = f(x) + \underbrace{0(x)}_{=\,0} = f(x)$, so A4.

For A5, define $(-f)(x) = -f(x)$. Then $f(x) + (-f)(x)$

$$= f(x) - f(x) = 0$$

$$= 0(x).$$

Ex: Let V denote ordered pairs (x, y) and define addition as in $\mathbb{R}^2$.

However define scalar multiplication by

$$a(x,y) = (ax, y).$$

Is V a vector space?

$2(1,1) = (2,1)$, but $2(1,1) = (1+1)(1,1) = 1(1,1) + 1(1,1)$
$= (1,1) + (1,1) = (2,2).$

So $(2,1) \neq (2,2)$ so it is <u>not</u> a vector space.

Properties of vector spaces

1.) If $\vec{v} + \vec{u} = \vec{v} + \vec{w}$

$$\Rightarrow (-\vec{v}) + (\vec{v} + \vec{u}) = (-\vec{v}) + (\vec{v} + \vec{w})$$

$$\Rightarrow (-\vec{v} + \vec{v}) + \vec{u} = (-\vec{v} + \vec{v}) + \vec{w}$$

$$\Rightarrow \vec{0} + \vec{u} = \vec{0} + \vec{w}$$

$$\Rightarrow \vec{u} = \vec{w} \quad \text{(cancellation property)}$$

2.) $0\vec{v} = \vec{0}$:

$$0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}$$

$$\Rightarrow \vec{0} + 0\vec{v} = 0\vec{v} + 0\vec{v}$$

$$\Rightarrow \vec{0} = 0\vec{v} \quad \text{by cancellation.}$$

3.) $a\vec{0} = \vec{0}$

4.) If $a\vec{v} = \vec{0}$ either $a = 0$ or $\vec{v} = \vec{0}$.

5.) $(-1)\vec{v} = -\vec{v}$:

$$\vec{v} + (-1)\vec{v} = (1-1)\vec{v} = 0\vec{v} = \vec{0} = \vec{v} + (-\vec{v})$$

So $(-1)\vec{v} = -\vec{v}$ by cancellation.

Practice problems: 6.1: 1, 2a-h, 4, 6bc, 14